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# Almost-commutative geometries beyond the standard model: III. Vector doublets

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## Abstract

We will present a new extension of the standard model of particle physics in its almost-commutative formulation. This extension has as its basis the algebra of the standard model with four summands (Iochum *et al* 2004 *J. Math. Phys.* **45** 5003 (*Preprint* hep-th/0312276), Jureit J-H and Stephan C 2005 *J. Math. Phys.* **46** 043512 (*Preprint* hep-th/0501134), Schücker T 2005 Krajewski diagrams and spin lifts *Preprint* hep-th/0501181, Jureit *et al* 2005 *J. Math. Phys.* **46** 072303 (*Preprint* hep-th/0503190), Jureit J-H and Stephan C 2006 On a classification of irreducible almost commutative geometries: IV (*Preprint* hep-th/0610040)), and enlarges only the particle content by an arbitrary number of generations of left–right symmetric doublets which couple vectorially to the  $U(1)_Y \times SU(2)_w$  subgroup of the standard model. As in the model presented in Stephan (2007 Almost-commutative geometries beyond the standard model: II. New Colours *Preprint* hep-th/0706.0595), which introduced particles with a new colour, grand unification is no longer required by the spectral action. The new model may also possess a candidate for dark matter in the hundred TeV mass range with neutrino-like cross section.

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## 1. Introduction

In this paper, we present an extension of the standard model by an arbitrary number of left–right symmetric doublets which couple vectorially to the  $U(1)_Y \times SU(2)_w$  subgroup of the standard model. We will call these particles for simplicity *vector doublets*. This extension is done within the framework of noncommutative geometry which was pioneered by Connes [1].

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It extends the standard model as presented in the recent formulation of almost-commutative geometry [2–4] which is a slight modification of the original formulation [5].

The model treated here is the third viable extension of the standard model, following the *AC* model [6] and a model which realises Okun’s  $\theta$ -particles [7] within almost-commutative geometry [8]. These extensions are very rare because the constraints on the models from the axioms of almost commutative geometry and the spectral action are severe. Nevertheless, at least the *AC* model exhibits a viable dark matter candidate [9]. This might also be true for the vector doublet model presented here.

As a basis for the vector doublet model we take the formulation of the standard model with four summands in the matrix algebra, which was found in the classification of almost-commutative geometries [10, 11]. The vector doublet model has many similarities to the  $\theta$ -model; notably the constraints on the gauge couplings of the model no longer resemble those of grand unified theories.

Adding vector doublets has a rather small effect on the Higgs mass, but lowers the cut-off scale of the spectral action considerably. This may provide a natural explanation for the possible masses of the vector doublets which are in the upper TeV scale. Here we will not consider mixing of the generations, but this should certainly be investigated more closely since it may provide a clue to the matter–antimatter asymmetry of the universe.

The paper is organized as follows. We first give the basic notions of a spectral triple, the main building block of noncommutative geometry. Then we quickly review how the Yang–Mills–Higgs model is obtained via the fluctuated Dirac operator and the spectral action. This account is far from exhaustive, and we refer to [4, 5, 12] for a detailed presentation.

For the vector doublets, the details of the spectral triple and the lift of the automorphisms are given. The Lagrangian and the constraints on the couplings are calculated, and we give a short summary of the mass splitting of the doublet components due to radiative corrections. With the help of the one-loop renormalization group equations, the Higgs boson mass and the cut-off scales are calculated for up to three generations of vector doublets.

## 2. Finite spectral triples

In this section, we will give the necessary basic definitions of almost commutative geometries from a particle physics point of view. For our calculations, only the finite part matters, so we restrict ourselves to real, finite spectral triples in *KO*-dimension 6:  $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \chi)$ . Note that in the literature before [2–4], the finite part of the spectral triple was considered to be of *KO*-dimension zero. The change in this algebraic dimension amounts to some sign changes, i.e. the commutator for the real structure and the chirality changes into an anti-commutator and the anti-particles have opposite chirality with respect to the particles.

### 2.1. Basic definitions

The algebra  $\mathcal{A}$  is a finite sum of matrix algebras  $\mathcal{A} = \bigoplus_{i=1}^N M_{n_i}(\mathbb{K}_i)$  with  $\mathbb{K}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}$ , where  $\mathbb{H}$  denotes the quaternions. A faithful representation  $\rho$  of  $\mathcal{A}$  is given on the finite dimensional Hilbert space  $\mathcal{H}$ . The Dirac operator  $\mathcal{D}$  is a self-adjoint operator on  $\mathcal{H}$  and plays the role of the fermionic mass matrix.  $J$  is an anti-unitary involution,  $J^2 = 1$ , and is interpreted as the charge conjugation operator of particle physics. The chirality  $\chi$  is a unitary involution,  $\chi^2 = 1$ , whose eigenstates with eigenvalue  $+1(-1)$  are interpreted as right (left) particle states and  $-1(+1)$  for right (left) anti-particle states. These operators are required to fulfil Connes’ axioms for spectral triples:

- $[J, \mathcal{D}] = \{J, \chi\} = \{\mathcal{D}, \chi\} = 0$ ,

$$[\chi, \rho(a)] = [\rho(a), J\rho(b)J^{-1}] = [[\mathcal{D}, \rho(a)], J\rho(b)J^{-1}] = 0, \forall a, b \in \mathcal{A}.$$

- The chirality can be written as a finite sum  $\chi = \sum_i \rho(a_i)J\rho(b_i)J^{-1}$ . This condition is called *orientability*.
- The intersection form  $\cap_{ij} := \text{tr}(\chi\rho(p_i)J\rho(p_j)J^{-1})$  is non-degenerate,  $\det \cap \neq 0$ .  $p_i$  are minimal rank projections in  $\mathcal{A}$ . This condition is called *Poincaré duality*.

Now the Hilbert space  $\mathcal{H}$  and the representation  $\rho$  are decomposed into left and right, particle and anti-particle spinors and representations respectively:

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c \quad \rho = \rho_L \oplus \rho_R \oplus \overline{\rho}_L^c \oplus \overline{\rho}_R^c.$$

In this representation, the Dirac operator has the form

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{\mathcal{M}} \\ 0 & 0 & \overline{\mathcal{M}}^* & 0 \end{pmatrix},$$

where  $\mathcal{M}$  is the fermionic mass matrix connecting the left- and the right-handed fermions.

Since the individual matrix algebras have only one fundamental representation for  $\mathbb{K} = \mathbb{R}, \mathbb{H}$  and two for  $\mathbb{K} = \mathbb{C}$  (the fundamental one and its complex conjugate),  $\rho$  may be written as a direct sum of these fundamental representations with multiplicities

$$\rho(\bigoplus_{i=1}^N a_i) := (\bigoplus_{i,j=1}^N a_i \otimes 1_{m_{ji}} \otimes 1_{(n_j)}) \oplus (\bigoplus_{i,j=1}^N 1_{(n_i)} \otimes 1_{m_{ji}} \otimes \overline{a_j}).$$

The first summand denotes the particle sector and the second the anti-particle sector. For the dimensions of the unity matrices, we have  $(n) = n$  for  $\mathbb{K} = \mathbb{R}, \mathbb{C}$  and  $(n) = 2n$  for  $\mathbb{K} = \mathbb{H}$  and the convention  $1_0 = 0$ . The multiplicities  $m_{ji}$  are non-negative integers. Acting with the real structure  $J$  on  $\rho$  permutes the main summands and complex conjugates them. It is also possible to write the chirality as a direct sum

$$\chi = (\bigoplus_{i,j=1}^N 1_{(n_i)} \otimes \chi_{ji} 1_{m_{ji}} \otimes 1_{(n_j)}) \oplus (\bigoplus_{i,j=1}^N 1_{(n_i)} \otimes (-\chi_{ji}) 1_{m_{ji}} \otimes 1_{(n_j)}),$$

where  $\chi_{ji} = \pm 1$  according to the previous convention on left (right) handed spinors. One can now define the multiplicity matrix  $\mu \in M_N(\mathbb{Z})$  such that  $\mu_{ji} := \chi_{ji} m_{ji}$ . This matrix is symmetric and decomposes into a particle and an anti-particle matrix, the second being just the particle matrix transposed,  $\mu = \mu_P + \mu_A = \mu_P - \mu_P^T$ . The intersection form of the Poincaré duality is now simply  $\cap = \mu - \mu^T$ , see [13]. Note that in contrast to the case of  $KO$ -dimension zero, the multiplicity matrix is now anti-symmetric rather than symmetric.

### 2.2. Obtaining the Yang–Mills–Higgs theory

To complete our short survey on the almost-commutative standard model, we will give a brief glimpse on how to construct the actual Yang–Mills–Higgs theory. We started out with the fixed (for convenience flat) Dirac operator of a four-dimensional spacetime with a fixed fermionic mass matrix. To generate curvature, we have to perform a general coordinate transformation and then fluctuate the Dirac operator. This can be achieved by lifting the automorphisms of the algebra to the Hilbert space, unitarily transforming the Dirac operator with the lifted automorphisms and then building linear combinations. Again we restrict ourselves to the finite case. Except for complex conjugation in  $M_n(\mathbb{C})$  and permutations of identical summands in the algebra  $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \dots \oplus \mathcal{A}_N$ , every algebra automorphism  $\sigma$  is inner,  $\sigma(a) = uau^{-1}$  for a unitary  $u \in U(\mathcal{A})$ . Therefore, the connected component of the automorphism group is  $\text{Aut}(\mathcal{A})^e = U(\mathcal{A})/(U(\mathcal{A}) \cap \text{Center}(\mathcal{A}))$ . Its lift to the Hilbert space [14]

$$\mathbb{L}(\sigma) = \rho(u)J\rho(u)J^{-1}$$

is multi-valued. To avoid the multi-valuedness in the fluctuations, we allow a central extension of the automorphism group.

The fluctuation  ${}^f\mathcal{D}$  of the Dirac operator  $\mathcal{D}$  is given by a finite collection  $f$  of real numbers  $r_j$  and algebra automorphisms  $\sigma_j \in \text{Aut}(\mathcal{A})^e$  such that

$${}^f\mathcal{D} := \sum_j r_j \mathbb{L}(\sigma_j) \mathcal{D} \mathbb{L}(\sigma_j)^{-1}, \quad r_j \in \mathbb{R}, \quad \sigma_j \in \text{Aut}(\mathcal{A})^e.$$

We consider only fluctuations with real coefficients since  ${}^f\mathcal{D}$  must remain self-adjoint. The sub-matrix of the fluctuated Dirac operator  ${}^f\mathcal{D}$ , which is equivalent to the mass matrix  $\mathcal{M}$ , is often denoted by  $\varphi$ , the ‘Higgs scalar’, in the physics literature. But one has to be careful, as will be shown below explicitly. It may happen that the lifted automorphisms commute with the initial Dirac operator and one finds  ${}^f\mathcal{D} = \sum_i r_i \mathcal{D}$  for the finite part of the spectral triple. This behaviour appeared for the first time in the electro-strong model in [11], where the fermions couple vectorially to all gauge groups and no Higgs field appears. In the model presented below, the spectral triple can be decomposed into a direct sum consisting of the standard model and two new particles. The initial Dirac operator of the new particles commutes with the corresponding part of the lift and thus does not participate in the Higgs mechanism.

An almost commutative geometry is the tensor product of a finite noncommutative triple with an infinite, commutative spectral triple. By Connes’ reconstruction theorem [15, 16] it is known that the latter comes from a Riemannian spin manifold, which will be taken to be any four-dimensional, compact manifold. The spectral action of this almost-commutative spectral triple is defined to be the number of eigenvalues of the Dirac operator  ${}^f\mathcal{D}$  up to a cut-off  $\Lambda$ . Via the heat-kernel expansion one finds, after a long and laborious calculation [4, 5], a Yang–Mills–Higgs action combined with the Einstein–Hilbert action and a cosmological constant:

$$\begin{aligned} S_{CC}[e, A_{L/R}, \varphi] &= \text{tr} \left[ h \left( \frac{{}^f\mathcal{D}^2}{\Lambda^2} \right) \right] \\ &= \int_M \left\{ \frac{2\Lambda_c}{16\pi G} - \frac{1}{16\pi G} R + a(5R^2 - 8R_{\mu\nu}R^{\mu\nu} - 7R_{\mu\nu\lambda\tau}R^{\mu\nu\lambda\tau}) \right. \\ &\quad + \sum_i \frac{1}{2g_i^2} \text{tr} F_{i\mu\nu}^* F_i^{\mu\nu} + \frac{1}{2} (D_\mu \varphi)^* D^\mu \varphi \\ &\quad \left. + \lambda \text{tr}(\varphi^* \varphi)^2 - \frac{1}{2} \mu^2 \text{tr}(\varphi^* \varphi) + \frac{1}{12} \text{tr}(\varphi^* \varphi) R \right\} dV + \mathcal{O}(\Lambda^{-2}), \end{aligned} \tag{2.1}$$

where  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a positive test function. The coupling constants are functions of the first moments  $h_0, h_2$  and  $h_4$  of the test function:

$$\begin{aligned} \Lambda_c &= \alpha_1 \frac{h_0}{h_2} \Lambda^2, & G &= \alpha_2 \frac{1}{h_2} \Lambda^{-2}, & a &= \alpha_3 h_4, \\ g_i^2 &= \alpha_{4i} \frac{1}{h_4}, & \lambda &= \alpha_5 \frac{1}{h_4}, & \mu^2 &= \alpha_5 \frac{h_2}{h_4} \Lambda^2. \end{aligned} \tag{2.2}$$

The curvature terms  $F_{\mu\nu}$  and the covariant derivative  $D_\mu$  are in the standard form of Yang–Mills–Higgs theory. The constants  $\alpha_j$  depend in general on the special choice of the matrix algebra and on the Hilbert space, i.e. on the particle content. For details of the computation, we refer to [4, 5].

This action is valid at the cut-off  $\Lambda$  where it ties together the coupling constants  $g_i$  of the gauge connections and the Higgs coupling  $\lambda$  since they originate from the same heat-kernel coefficient. For the standard model with three generations the calculation of the gauge

couplings in (2.2) imposes at  $\Lambda$  conditions on the  $U(1)_Y$ ,  $SU(2)_w$  and  $SU(3)_c$  couplings  $g_1$ ,  $g_2$  and  $g_3$ , respectively, comparable to those of grand unified theories:

$$5g_1^2 = 3g_2^2 = 3g_3^2. \tag{2.3}$$

But since the lift of the automorphisms produces extra free parameters through the  $U(1)$  central charges the first equality can always be modified by a different choice of the central charge. Therefore, only the gauge couplings of noncommutative gauge groups underlie constraints from the spectral action.

In the same way as for the gauge couplings, the spectral action also implies constraints for the quartic Higgs coupling  $\lambda$  and the Yukawa couplings. The full set of constraints for the standard model reads [2, 4, 17]

$$3g_2^2 = 3g_3^2 = 3\frac{Y_2^2}{H} \frac{\lambda}{24} = \frac{3}{4}Y_2. \tag{2.4}$$

Here  $Y_2$  is the sum of all Yukawa couplings  $g_f$  squared and  $H$  is the sum of all Yukawa couplings raised to the fourth power. Our normalizations are  $m_f = \sqrt{2}(g_f/g_2)m_W, (1/2)(\partial\phi)^2 + (\lambda/24)\phi^4$ .

As we will see in the following, the grand unified constraint  $g_2^2 = g_3^2$  at the cut-off  $\Lambda$  is a very special case. It is valid for the standard model, but in general it will not hold. The model presented in this paper is one more example for different constraints for  $g_2$  and  $g_3$  at the cut-off energy. For possible extensions of the standard model within the framework of almost-commutative geometry, these constraints may limit the particle content in a crucial way.

### 3. The spectral triple

The model presented here is based on the spectral triple of the standard model with four summands in the matrix algebra [11]. In contrast to previous extensions of the standard model [6, 8], the algebra is not enlarged:

$$\mathcal{A} = \mathcal{A}_{SM} = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \ni (a, b, c, d). \tag{3.1}$$

Instead, we enlarge the standard model by adding an *a priori* arbitrary number of generations of  $SU(2)_w$  vector doublets. As we will see later, anomaly cancellation also requires vector-like hypercharges. The representation of the algebra for these new particles is

$$\rho_L(d) = d1_2, \quad \rho_R(b) = \bar{b}1_2, \quad \rho_L^c(a) = a, \quad \rho_R^c(a) = a. \tag{3.2}$$

One sees immediately the vector-like coupling to the quaternion sub-algebra in the anti-particle part of the representation. This results in a vector-like coupling to the  $SU(2)_w$  subgroup of the standard model. Requiring the model to be anomaly free will induce the vector-like hypercharge coupling. Note that these vector doublets do not satisfy all the physical requirements which had been put forward in [11] to classify almost-commutative geometries. Notably the requirement of an unbroken colour group is not satisfied since the  $SU(2)_w$  subgroup acts as a colour for the vector doublets and is broken by the Higgs mechanism.

The complete representation for the model is the direct sum of the standard model representation and the representation for the vector doublets:

$$\rho = \rho_{SM} \oplus \rho_{vec} \quad \text{with} \quad \rho_{vec}(a, b, d) = \rho_L(d) \oplus \rho_R(b) \oplus \rho_L^c(a) \oplus \rho_R^c(a). \tag{3.3}$$

The same holds for the Hilbert space,  $\mathcal{H} = \mathcal{H}_{SM} \oplus \mathcal{H}_{new}$ . For  $N$  generations of vector doublets, their Hilbert space is

$$\mathcal{H}_{vec} = (\mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}^N)_L \oplus (\mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}^N)_R \oplus \text{anti-particles}. \tag{3.4}$$

The dimension of  $\mathcal{H}_{\text{vec}}$  depends on the number of generations  $N$  of vector doublets and reads  $\dim(\mathcal{H}_{\text{vec}}) = 8N$ .

We will denote the spinors of the vector doublets  $\psi_1$  and  $\psi_2$  which are both hypercharge singlets

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_L \oplus \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_R \oplus \begin{pmatrix} \psi_1^c \\ \psi_2^c \end{pmatrix}_L \oplus \begin{pmatrix} \psi_1^c \\ \psi_2^c \end{pmatrix}_R \in \mathcal{H}_{\text{vec}}. \quad (3.5)$$

The Dirac operator contains the masses of the vector doublets and a CKM-like matrix which mixes the generations in the case of  $N \geq 2$ . For a first analysis of the model, we will consider the CKM-like matrix to be the unity matrix. A nontrivial mixing matrix may be interesting when considering leptogenesis-like processes to explain the particle–anti-particle asymmetry in the universe.

The Dirac operator for one generation of vector doublets is

$$D_{\text{vec}} = \begin{pmatrix} 0 & \mathcal{M}_{\text{vec}} & 0 & 0 \\ \mathcal{M}_{\text{vec}}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{\mathcal{M}}_{\text{vec}} \\ 0 & 0 & \overline{\mathcal{M}}_{\text{vec}}^* & 0 \end{pmatrix} \quad \text{with} \quad \mathcal{M}_{\text{vec}} = m_\psi 1_2. \quad (3.6)$$

From the Krajewski diagram, figure 1 in the appendix, it is straightforward to see that all the axioms for the spectral triple are fulfilled. The second possibility of realising vector doublets is depicted in the second Krajewski diagram, figure 2 in the appendix. This model exhibits essentially the same features as those presented above; only the sign of the hypercharges for the vector doublets is reversed.

#### 4. The gauge group, the lift and the constraints

The automorphisms that have to be lifted coincide with the group of unitaries of the noncommutative part of the algebra [12]:

$$\mathcal{U}^{nc}(\mathcal{A}) = SU(2)_w \times U(3) \ni (v, w). \quad (4.1)$$

Defining  $u := \det(w) \in U(1)$ , the particle part of the lift decomposes into a left-handed part and a right-handed part:

$$\mathbb{L}(v, u^{p_1}, u^{p_2} w, u^{p_3}) = \mathbb{L}_L(v, u^{p_1}, u^{p_2} w) \oplus \mathbb{L}_R(v, u^{p_2} w, u^{p_3}) \quad (4.2)$$

with  $p_i, q_i \in \mathbb{Z}$ . We will impose here that the standard model remains unchanged, i.e. that all the charges are the standard ones. From the standard model, part of the lift then follows  $p_1 = -p_3 = -1/2$  and  $p_2 = 1/6 - 1/3$  through the requirement of anomaly cancellation. This reduces  $U(3)$  to  $U(1)_Y \times SU(3)_c$  in the correct representation.

The exact form of the lift for the vector doublets is given by

$$\mathbb{L}_{\text{vec}}(v, u^{p_1}, u^{p_3}) = \text{diag}[u^{p_1} v, u^{-p_3} v], \quad (4.3)$$

which is automatically anomaly free. We now see that the hypercharges of the vector doublets have been determined by fixing the hypercharges of the standard model. Therefore, the almost-commutative version of vector doublets is far more constrained than vector doublets in the general Yang–Mills–Higgs setting where the hypercharges are free parameters.

One immediately sees that the vector doublets have the same charge assignment as the left-handed electron-neutrino doublet. Therefore, the electro-magnetic charge of the components of the vector doublets are  $Q_{el} = -e$  for  $\psi_1$  and  $Q_{el} = 0$  for  $\psi_2$ . We will henceforth call  $\psi_1 =: \psi^-$  and  $\psi_2 =: \psi^0$ . This charge assignment is summarized in table 1. For the model derived from the Krajewski diagram in figure 2, one finds a hypercharge charge assignment

**Table 1.** Charge assignment for a negatively charged component.

	I	$I_3$	$Y_{\text{vec}}$	$Q_{el}$
$(\psi_1)_{L,R} = \psi_{L,R}^-$	2	$-\frac{1}{2}$	$-\frac{1}{2}$	$-e$
$(\psi_2)_{L,R} = \psi_{L,R}^0$	2	$+\frac{1}{2}$	$-\frac{1}{2}$	0

**Table 2.** Charge assignment for a positively charged component.

	I	$I_3$	$Y_{\text{vec}}$	$Q_{el}$
$(\psi_1)_{L,R} = \psi_{L,R}^0$	2	$-\frac{1}{2}$	$+\frac{1}{2}$	0
$(\psi_2)_{L,R} = \psi_{L,R}^+$	2	$+\frac{1}{2}$	$+\frac{1}{2}$	$+e$

with opposite signs and therefore opposite electro-magnetic charges; see table 2. We will henceforth concentrate on the first case with  $\psi^-$  and  $\psi^0$ .

Since the vector doublets couple has vector-like couplings to the gauge group, the mass matrix  $\mathcal{M}_{\text{new}}$  commutes with the lift  $\mathbb{L}_{\text{new}}$  and it follows from the the inner fluctuations that the masses have no connection to the standard model Higgs field

$$\sum_j r_j \mathbb{L}_{L,\text{vec}} \mathcal{M}_{\text{vec}} \mathbb{L}_{R,\text{vec}}^{-1} = \sum_j r_j \mathcal{M}_{\text{vec}} =: M_{\text{vec}}. \quad (4.4)$$

Therefore,  $M_{\text{vec}}$  contains the gauge invariant masses of the vector doublets where the real numbers  $r_i$  are determined by the standard model part. This phenomenon of gauge invariant masses in almost-commutative geometry appeared first in the case of the electro-strong model [11]. It also appears in the AC model [6] and in the standard model with Majorana neutrinos [4].

From the spectral action, one now immediately obtains the Lagrangian for the new particles:

$$\begin{aligned} \mathcal{L}_{\text{vec}} = & +i \sum_{i=1,\dots,N} (\bar{\psi}^-, \bar{\psi}^0)_L^i D^\psi \begin{pmatrix} \psi^- \\ \psi^0 \end{pmatrix}_L^i + i \sum_{i=1,\dots,N} (\bar{\psi}^-, \bar{\psi}^0)_R^i D^\psi \begin{pmatrix} \psi^- \\ \psi^0 \end{pmatrix}_R^i \\ & - \sum_{i=1,\dots,N} (\bar{\psi}^-, \bar{\psi}^0)_L^i M_{\text{vec}}^i \begin{pmatrix} \psi^- \\ \psi^0 \end{pmatrix}_R^i + \text{Hermitian conjugate}, \end{aligned} \quad (4.5)$$

where the covariant derivatives are given by

$$D^\psi = \partial_\mu + ig_1 \frac{Y_{\text{vec}}}{2} B_\mu + ig_2 W_\mu^k \frac{\tau_k}{2}. \quad (4.6)$$

Here,  $g_1$  and  $g_2$  are the standard model  $U(1)_Y$  and  $SU(2)_w$  gauge couplings with their corresponding generators respectively.

From the spectral action, it is now easy to calculate the constraints on the gauge couplings, the quartic Higgs coupling and the Yukawa couplings at the cut-off  $\Lambda$ . The normalization of the quartic Higgs coupling is taken to be the same as for the standard model. Then the new constraints read

$$\left(3 + \frac{N}{2}\right) g_2^2 = 3g_3^2 = 3 \frac{Y_2^2}{H} \frac{\lambda}{24} = \frac{3}{4} Y_2. \quad (4.7)$$

$Y_2$  and  $H$  include the Yukawa couplings of the standard model including a large Yukawa coupling for the  $\tau$ -neutrino [4]. We do not have any constraints on  $g_1$  since the central charges that enter through the lift are free parameters.



This model has again the feature that models beyond the standard model in almost-commutative geometry will in general not exhibit the constraint  $g_2 = g_3$  from grand unified theories. A similar constraint as in (2.4) already appeared in the model presented in [8].

In the following, we will neglect all standard model Yukawa couplings except the top quark coupling  $g_t$  and the  $\tau$ -neutrino Yukawa coupling  $g_\nu$  which are adjusted to reproduce the correct top quark mass [4]. We will not go into the details of the seesaw mechanism in almost-commutative geometry, but refer to [4, 18]. Under these assumptions, the relevant constraints on the couplings at the cut-off  $\Lambda$  read

$$g_3^2 = \left(1 + \frac{N}{6}\right) g_2^2 \quad (4.8)$$

$$g_t^2 = \frac{4 + \frac{4N}{6}}{3 + R^2} g_2^2 \quad \text{with} \quad R := \frac{g_\nu}{g_t} \quad (4.9)$$

$$\lambda = 8 \left(3 + \frac{N}{2}\right) \frac{3 + R^4}{(3 + R^2)^2} g_2^2 \quad (4.10)$$

The ratio  $R$  of the Yukawa coupling  $g_\nu$  of the  $\tau$ -neutrino and the top quark Yukawa coupling  $g_t$  is fixed by the requirement that the renormalization group flow produces the measured pole mass of the top quark,  $m_t = 170.9 \pm 1.8$  GeV [25].

## 5. Radiative corrections to the vector doublet masses

The masses of the two components  $\psi^-$  and  $\psi^0$  of the vector doublet are *a priori* degenerate,  $m_{\psi^-} = m_{\psi^0} = m_\psi$ . But this degeneracy will split due to radiative corrections for energies below the mass of the Z-boson  $m_Z$ . For a detailed phenomenological discussion, see [19].

The calculations are well known, and we will only give the result for the mass splitting. Defining  $r = \left(\frac{m_\psi}{m_Z}\right)^2$ , one finds for the mass difference

$$\Delta m_\psi = \frac{\alpha}{2} M_Z f(r) \quad \text{with} \quad f(r) = \int_0^1 (2-x) \ln \left(1 + \frac{x}{(1-x)^2 r}\right), \quad (5.1)$$

where the charged particle is heavier than its neutral partner,  $m_{\psi^-} = m_{\psi^0} + \Delta m_\psi$ . Taking the limit  $m_\psi \gg m_Z$ , i.e.  $r \gg 1$ , which will be interesting from the dark matter point of view, one finds the asymptotic mass difference  $\Delta m_\psi = \frac{1}{2} \alpha M_Z \simeq 355$  MeV. It is also interesting to note that the lifetime of the charged particle is rather short with 0.5 to 2 ns [20].

Since there are no terms in the Lagrangian coupling the vector bosons to standard model particles, we will consider the neutral particle as stable. It behaves essentially like a neutrino and its spin-independent cross section is of the same order,  $\sigma_{\text{si}} \sim 10^{-39}$  cm<sup>2</sup>. If these particles are heavy enough  $m_\psi > 10$  TeV, they can escape direct detection since current experiments are not sensitive for ultramassive dark matter particles. Below  $\sim 10$  TeV, vector doublets can be excluded as a dark matter candidate [21]. It has also been shown that neutrino-like particles with masses from 250 TeV to 550 TeV may saturate the dark matter abundance of the universe [22]. If the vector doublets are heavier than 550 TeV, they will over-close the universe. We will, therefore, concentrate on the mass region between 10 TeV and 550 TeV.

## 6. The renormalization group equations

We will now give the one-loop  $\beta$ -functions of the standard model with three generations of standard model particles,  $N$  generations of vector doublets. They will serve to evolve the

constraints (4.7) from  $E = \Lambda$  down to our energies  $E = m_Z$ . We set:  $t := \ln(E/m_Z)$ ,  $dg/dt =: \beta_g$ ,  $\kappa := (4\pi)^{-2}$ . We will neglect the running of the gauge invariant masses of the vector doublets and treat them as free parameters. Furthermore, all threshold effects will be neglected.

The  $\beta$ -functions for the standard model with  $N$  generations of vector doublets are [23, 24]

$$\beta_{g_i} = \kappa b_i g_i^3, \quad b_i = \left( \frac{41}{6} + \frac{2}{3}N, -\frac{19}{6} + \frac{2}{3}N, -7 \right), \quad (6.1)$$

$$\beta_t = \kappa \left[ -\sum_i c_i^u g_i^2 + Y_2 + \frac{3}{2}g_t^2 \right] g_t, \quad (6.2)$$

$$\beta_\lambda = \kappa \left[ \frac{9}{4}(g_1^4 + 2g_1^2 g_2^2 + 3g_2^4) - (3g_1^2 + 9g_2^2)\lambda + 4Y_2\lambda - 12H + 4\lambda^2 \right], \quad (6.3)$$

with

$$c_i^t = \left( \frac{17}{12}, \frac{9}{4}, 8 \right), \quad Y_2 = 3g_t^2, \quad H = 3g_t^4. \quad (6.4)$$

The Yukawa coupling of the  $\tau$ -neutrino can be neglected in the evolution of the renormalization group equations, since the seesaw mechanism renders it small compared to the top quark Yukawa coupling.

The gauge couplings decouple from the other equations and have identical evolutions in both energy domains:

$$g_i(t) = g_{i0} / \sqrt{1 - 2\kappa b_i g_{i0}^2 t}. \quad (6.5)$$

The initial conditions are taken from experiment [25]:

$$g_{10} = 0.3575, \quad g_{20} = 0.6514, \quad g_{30} = 1.221. \quad (6.6)$$

Then the unification scale  $\Lambda$  is the solution of  $(1 + \frac{N}{6})g_2^2(\ln(\Lambda/m_Z)) = g_3^2(\ln(\Lambda/m_Z))$ :

$$\Lambda = m_Z \exp \frac{g_{20}^{-2} - (1 + \frac{N}{6})g_{30}^{-2}}{2\kappa(b_2 - (1 + \frac{N}{6})b_3)}, \quad (6.7)$$

and depends on the number of generations of vector doublets  $N$ .

## 7. The Higgs boson mass

The aim is now to calculate the mass of the Higgs boson,  $m_H$ , fixing the quartic coupling  $\lambda$  at the cut-off  $\Lambda$  and then evolving it down to the pole mass with the renormalization group equations. We require that all couplings remain perturbative and we obtain the pole masses of the Higgs boson and the top quark:

$$m_H^2 = \frac{4}{3} \frac{\lambda(m_H)}{g_2(m_Z)^2} m_W^2, \quad m_t = \sqrt{2} \frac{g_t(m_t)}{g_2(m_t)} m_W. \quad (7.1)$$

As experimental input we have the initial conditions of the three standard model gauge couplings (6.6) and the mass of the top quark,  $m_t = 170.9 \pm 1.8$  GeV [25]. As mentioned before, the masses of the vector doublets are taken to be between 10 TeV and 550 TeV. We will calculate the mass of the Higgs boson for these two extreme values with the constraints (4.8) to (4.10).

For the pure standard model, we find a Higgs mass of  $m_H = 167.8_{-1.7}^{+1.8}$  GeV and a cut-off of  $\Lambda = 1.1 \times 10^{17}$  GeV. This is in good agreement with previous calculations [4, 18]. We

**Table 3.** One generation of vector doublets.

One generation	$m_H$	$\Lambda$
$m_\psi = 10 \text{ TeV}$	$178, 7^{+0,7}_{-0,7} \text{ GeV}$	$5.3 \times 10^{11} \text{ GeV}$
$m_\psi = 550 \text{ TeV}$	$177, 9^{+0,8}_{-0,7} \text{ GeV}$	$8.5 \times 10^{11} \text{ GeV}$

**Table 4.** Two generations of vector doublets.

Two generations	$m_H$	$\Lambda$
$m_\psi = 10 \text{ TeV}$	$191, 4^{+0,3}_{-0,2} \text{ GeV}$	$10^9 \text{ GeV}$
$m_\psi = 550 \text{ TeV}$	$189, 3^{+0,3}_{-0,2} \text{ GeV}$	$2.1 \times 10^9 \text{ GeV}$

**Table 5.** Three generations of vector doublets.

Three generations	$m_H$	$\Lambda$
$m_\psi = 10 \text{ TeV}$	$204, 9^{+0,3}_{-0,3} \text{ GeV}$	$2, 8 \times 10^7 \text{ GeV}$
$m_\psi = 550 \text{ TeV}$	$201, 0^{+0,2}_{-0,1} \text{ GeV}$	$5, 4 \times 10^7 \text{ GeV}$

**Table 6.** Five generations of vector doublets.

Five generations	$m_H$	$\Lambda$
$m_\psi = 10 \text{ TeV}$	$233, 1^{+0,9}_{-0,9} \text{ GeV}$	$2, 9 \times 10^5 \text{ GeV}$
$m_\psi = 550 \text{ TeV}$	$224, 4^{+0,5}_{-0,7} \text{ GeV}$	$8, 0 \times 10^5 \text{ GeV}$

will now add subsequently one, two and three generations of vector doublets to the standard model. To simplify the analysis we will assume the masses of the vector doublets to be equal and the CKM mixing matrix to be trivial, i.e. the unity matrix. Nontrivial mixing between the generations may perhaps be interesting when considering the particle–anti-particle asymmetry in cosmology. Furthermore we will restrict ourselves to the two extrema of the possible masses for the vector doublets:  $10 \text{ TeV} \leq m_\psi \leq 550 \text{ TeV}$ .

For the case of one generation of vector doublets, the Higgs mass and the cut-off scale are summarized in table 3. Note that the cut-off scale is lowered considerably, by six orders of magnitude with respect to the pure standard model. This phenomenon has two origins. On one hand the running of the  $SU(2)_w$  coupling  $g_2$  is diminished due to the presence of the vector doublets, while the running of the  $SU(3)_c$  coupling  $g_3$  remains unchanged since the vector doublets are colour singlets. Secondly, the constraint (4.8) on  $g_2$  and  $g_3$  at the cut-off gets modified. The effect of the vector doublets on the running of the couplings is rather small compared to their effect on the constraint (4.8).

For two generations of vector doublets, the Higgs boson mass and cut-off scale are summarized in table 4. One notes that the influence of the vector doublets on the Higgs mass is rather small. This should be compared to other models beyond the standard model [8], which can increase the Higgs mass by up to  $\sim 160 \text{ GeV}$  to  $m_{\text{Higgs}} \sim 380 \text{ GeV}$ . Table 5 shows the Higgs boson masses and the cut-off scales for three generations of vector doublets. To underline the general behaviour we also give a more extreme case with five generations of vector doublets, see table 6. Here the cut-off scale has dropped down to the order of the

vector doublet masses. This is certainly a very interesting feature since it would give a natural explanation for the mass scale of the vector doublets. Furthermore, the Higgs mass is raised by  $\sim 65$  GeV with respect to the pure standard model. This allows the model to be clearly distinguished from the almost-commutative standard model by the LHC. The signature for this model would then be a very heavy Higgs boson and no further particles, since the masses of vector doublets should be above the energy achieved by the LHC.

## 8. Conclusions and outlook

We have presented a particle model based on an almost-commutative geometry which contains the standard model as a sub-model. The spectral triple of the standard model is modified only slightly, in the sense that the matrix algebra of the standard model stays unchanged and only an arbitrary number of  $SU(2)_w$  vector doublets are added.

These vector doublets are anomaly free, but their hypercharges are constrained by the standard model hypercharges. This results in an electro-magnetically charged component of the doublet with one electron charge and a neutral component. Here again almost-commutative geometry is far more restrictive than general Yang–Mills–Higgs theory where in principle any hypercharge for vector doublets is allowed, and therefore both components of the doublets may be charged. The masses of the vector doublets are gauge invariant, i.e. they do not couple to the Higgs boson. Furthermore, the new particles are colour singlets with respect to the  $SU(3)_c$  colour group of the standard model.

The neutral particle in the doublet has a slightly lower mass than the charged particle with a mass difference of  $\Delta m_\psi \sim 350$  MeV. This allows the charged particle to decay into its stable, neutral partner. The spin-independent cross section of the neutral particle is of the same order of magnitude as a neutrino's cross section,  $\sigma_{\text{si}} \sim 10^{-39}$  cm<sup>2</sup>. If these particles are sufficiently heavy, they may be interesting dark matter candidates.

Considering masses for the vector doublets between 10 TeV and 550 TeV one finds, when adding up to three generations to the standard model, only a slight effect of  $\sim 35$  GeV on the Higgs mass. In contrast, the cut-off scale decreases considerably down to  $\sim 10^7$  GeV for three generations of new particles. This low cut-off scale could explain in a very natural way the scale of the gauge invariant masses of the vector doublets.

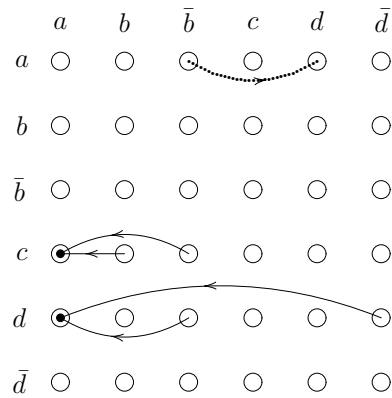
Concluding, one can certainly say that this model seems to be an interesting and promising extension of the standard model. Open issues are the direct detectability of extremely heavy vector doublets by experiments such as EDELWEIS and the effect of a nontrivial CKM-like mixing matrix.

## Acknowledgments

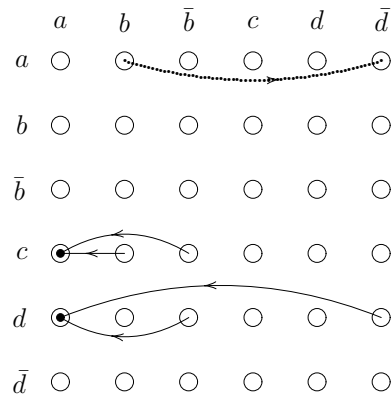
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## Appendix. The Krajewski diagram

In this appendix, we present the Krajewski diagrams which were used to construct the model treated in this publication. Krajewski diagrams do for spectral triples what the Dynkin and weight diagrams do for groups and representations. For an introduction into the formalism of Krajewski, we refer to [11, 13]. The Krajewski diagram for the model presented in this paper



**Figure 1.** Krajewski diagram of the standard model with right-handed neutrino. The anti-particle part and the arrow representing Majorana masses has not been drawn. The vector doublets are depicted by the dotted line.



**Figure 2.** Krajewski diagram for the vector doublet model with reversed hypercharges.

is depicted in figure 1. It shows one generation of standard model particles and one generation of vector doublets.

The arrows encoding the new particles are drawn on the  $a$ -line. Note the similarity to the standard model quark sector which sits on the  $c$ -line.

The multiplicity matrix  $\mu$  associated with the Krajewski diagram in figure 1, with three generations of standard model particles and  $N$  generations of vector doublets, can be directly read off to be

$$\mu = \begin{pmatrix} 0 & N & 0 & -N \\ 0 & 0 & 0 & 0 \\ -3 & 6 & 0 & 0 \\ -3 & 3 & 0 & 3 \end{pmatrix}. \tag{A.1}$$

The axiom of the Poincaré duality is fulfilled since  $\det(\mu - \mu^t) = 36N^2 - 108N + 81 \neq 0$  for all  $N \in \mathbb{N}$ . Only the right-handed neutrinos violate the axiom of orientability [26], which is also the case for the pure standard model. It is also possible to reverse the arrow of the new

particles, exchanging right-handed and left-handed vector doublets. But this does not change the general result.

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